

Spectral geometry for strings and branes

Dmitri V. Vassilevich

Institut für Theoretische Physik, Universität Leipzig,
Augustusplatz 10/11, D-04109 Leipzig, Germany
and

V. A. Fock Institute of Physics, St.Petersburg University,
198904 St. Petersburg, Russia*

July 11, 2001

Abstract

I give a short guide into applications of the heat kernel technique to the string/brane physics with an emphasis on the emerging boundary value problems.

1 Introduction

The heat kernel technique is especially effective for evaluation of the one-loop beta functions, quantum anomalies and low-energy effective action. These three topics cover a large part of calculations in the sigma model approach to string theory. Local geometry of the closed strings is too simple to reveal advantages of the heat kernel methods. For open strings power of these methods was recognized long ago (see e.g. [1]). Now, boundary conditions and boundary dynamics are of primary interest in string/brane theory.

In this note I give a short review of boundary value problems which appear in various string and brane models, list known properties of these problems and present some open questions. We will mostly deal with partial differential operators of Laplace type acting on smooth sections of a vector bundle. Any operator of this type can be uniquely represented as

$$D = -(\nabla^\mu \nabla_\mu + E(x)) \quad (1)$$

with an appropriate connection ∇ and an endomorphism (matrix-valued function) E . Usually, but not always, there is a power law asymptotics of the heat

*E.mail: Dmitri.Vassilevich@itp.uni-leipzig.de

kernel as $t \rightarrow +0$

$$\text{Tr}(f e^{-tD}) \simeq \sum_{n=0} t^{(n-m)/2} a_n(f, D), \quad (2)$$

where m is dimension of the manifold, and the heat kernel coefficients are a_n can be represented through bulk and surface integrals.

Actual calculations of the coefficients a_n are not easy. However, these calculation are not as hard as one can imagine looking at (very) long resulting expressions. For manifolds with boundaries the general idea of the most powerful (functorial) method can be found in [2] (with some minor corrections in [3]). The heat kernel coefficients for Dirichlet, Robin and mixed boundary conditions up to a_5 can be found in [4].

This paper is organized as follows. The next section is devoted to open strings, Dirichlet branes and some more general objects. Non-local mixtures (bound states) of strings and branes are described in sec. 3. In sec. 4 I comment on the heat kernel expansion for singular geometries of the brane-world type. Sec. 5 contains concluding remarks.

2 Open strings and Dirichlet branes

If we adopt Euclidean signature on both the target space and the world sheet \mathcal{M} and neglect dimensional couplings, the action for the open string sigma model takes the form:

$$\begin{aligned} S = \int_{\mathcal{M}} d^2 z \sqrt{h} & \left(\frac{1}{2} G_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu \right. \\ & \left. + \frac{1}{2\sqrt{h}} \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \\ & + \int_{\partial\mathcal{M}} d\tau (A_\mu \partial_\tau X^\mu). \end{aligned} \quad (3)$$

Here we adopted the standard notations of [5]. Let us split the string coordinate $X = \bar{X} + \xi$ into background part \bar{X} and fluctuations ξ . Keeping the second order terms in ξ which are relevant for the one-loop corrections we obtain:

$$\begin{aligned} S_2 = \frac{1}{2} \int_{\mathcal{M}} d^2 z \sqrt{h} \xi D \xi \\ + \frac{1}{2} \int_{\partial\mathcal{M}} d\tau \xi \left(\nabla_N + \frac{1}{2} (\nabla_\tau \Gamma + \Gamma \nabla_\tau) + \mathcal{S} \right) \xi. \end{aligned} \quad (4)$$

The operator D , the covariant derivative ∇ and the matrix-valued functions Γ and \mathcal{S} depend on the background fields $A(\bar{X})$, $B(\bar{X})$, $G(\bar{X})$. The boundary conditions should satisfy two important requirements: i) the operator D should

be formally self-adjoint (symmetric); ii) the boundary term in (4) should vanish. There are two obvious local choices of such boundary conditions:

$$\left(\nabla_N + \frac{1}{2}(\nabla_\tau \Gamma + \Gamma \nabla_\tau) + \mathcal{S} \right) \xi|_{\partial\mathcal{M}} = 0 \quad (5)$$

corresponding to open strings, and

$$\xi|_{\partial\mathcal{M}} = 0 \quad (6)$$

describing the Dirichlet branes. Generically, some of the components of ξ satisfy (5) while the other satisfy (6).

The boundary conditions (5) contain both normal and tangential derivatives. Therefore, there are crucial differences from ordinary Neumann or Robin boundary value problem. In some literature the boundary conditions of the type (5) are called oblique. Study of corresponding boundary value problem has been initiated by Grubb [6] and then continued in [7, 8, 9, 10]. Let us briefly summarize the relevant results. On manifolds with boundary to ensure normal properties of the spectrum of an operator of Laplace type (as e.g. no more than finite number of negative modes) the boundary value problem must satisfy the so-called strong ellipticity condition. In the present context it requires that norm Γ is not too large (meaning that the gauge field strength is less than its critical value)¹. If the boundary value problem is strongly elliptic, there is an asymptotic expansion (2). Coefficients a_n with $n \leq 3$ for oblique boundary conditions have been calculated in [8, 9, 10].

The analysis [11] of the conformal invariance of open strings performed with the help of the heat kernel expansion confirms, in general, the earlier results [12]. However, some additional terms in the conformal variations appear [11] (see also [13]). Interpretation of these terms is still unclear.

In [14] we argued that multiplicative renormalizability of the open string sigma model suggests introduction in the action (3) a new surface coupling:

$$- \int_{\partial\mathcal{M}} V_\mu(X) \partial_N X^\mu. \quad (7)$$

Such coupling in fact appeared already in [15]. Acting as before, we arrive at the following boundary conditions:

$$(-(1 + \Lambda)\nabla_N \xi + L\xi)|_{\partial\mathcal{M}} = 0, \quad (8)$$

where the operator L contains tangential derivatives only, the matrix Λ depends on V : $\Lambda_{\mu\nu} = D_\mu V_\nu$. Since the operator L is defined by the quadratic form of the boundary action, it is fixed up to an anti-hermitian part (which vanishes being sandwiched between two ξ). This ambiguity is to be used to make the operator $(1 + \Lambda)^{-1}L$ a hermitian operator on the boundary (to ensure hermiticity

¹Note, that Dirichlet and Robin boundary value problems are always strongly elliptic.

of the volume operator D). Solution for L has a rather complicated form. Consequently, the heat kernel calculations also become complicated. Presently, the heat kernel coefficients are known in few first orders of Λ or, equivalently, of V [14].

Increasing complexity of the calculations was not our motivation. The boundary conditions (8) possess a very interesting qualitative property. They interpolate between the open string boundary conditions $\Lambda = 0$ and the D-brane boundary conditions $\Lambda = -1$. This suggests a promising mechanism of spontaneous Dirichlet brane creation. Actual realization of this mechanism requires more effective methods of calculation of the beta-functions or of the heat kernel expansion.

3 Spectral branes

As has been already mentioned in the previous section, a consistent string sigma model is described by a mixture of the boundary conditions (5) and (6). For the Dirichlet branes such a mixture is prepared in a particularly simple way: it is enough to say that the $p + 1$ components satisfy (5) while the rest of the components obey (6). This picture can be easily visualized, but it is not unique. Projector on the Dirichlet components may be as well a complicated non-local operator. An example of such non-local construction was given in [16]. It uses the famous spectral boundary conditions of Atiyah, Patodi and Singer [17]. Such boundary conditions are defined with respect to a Dirac operator. We may take a “square root” P of the Laplace operator D appearing in the quadratic form of the action (4) defined as $P^\dagger P = D$ (if such a “square root” exists). To this end we have to find a representation of the Clifford algebra in two dimensions with the target space indices. Effectively, one has to decompose the target space into a direct sum of two-dimensional subspaces and define standard γ -matrices on each subspace. Consider the simplest case when D is just the free Laplacian. Then $P = \gamma^a \partial_a$. Define $\mathcal{H} = \gamma_\sigma \gamma^\tau \partial_\tau$, where τ is a geodesic coordinate on the boundary, σ is a normal coordinate. Now, we may impose the boundary conditions (5) and (6) independently on each eigenmode of \mathcal{H} . For example, we may say that the modes with positive eigenvalues of \mathcal{H} satisfy (5), and the rest obey (6). This is the essence of the S -brane construction. This object has a lot of interesting properties [16]. Most remarkable is the behavior of the S -branes under the T-duality transformation (see, e.g., [18] for a path integral derivation of the T-duality). S -branes may be mapped to themselves, to open strings and to D -branes. Therefore, S -branes play a rather special role in string physics. However, many points still have to be clarified. This includes classification of the Laplace operators which are representable through the Dirac operators. Heat trace asymptotics for these rather complicated boundary value problem have not been calculated so far.

4 Brane-world

The brane-world scenario [20] suggests a rather unusual type of the “boundary value” problem. It involves a singular surface Σ such that only the leading symbol of the operator D (i.e., the Riemann metric) is assumed to be continuous across Σ . All other geometric quantities including derivatives of the metric can jump on Σ . One has to assume some matching conditions on the limiting values of the field and of its normal derivative. Until very recent time the heat kernel expansion for that kind of problems escaped systematic study. As far as the standard brane world scenario is concerned, the heat kernel asymptotics can be found in [19, 21]. However, calculations for the most general matching conditions [22] are yet to be performed.

5 Conclusions

We have described several boundary value problems which appear in the string or brane context. As usually, one needs the β -functions and the effective action in the form of an expansion in the background fields. These problems can be solved by the heat kernel methods.

A non-standard application of the heat kernel technique motivated by strings is duality relations between functional determinants (see e.g. [23]). The simplest example of such relations is transformation of the effective action $W[\phi] = -\frac{1}{2} \log \det(-e^\phi \nabla_\mu e^{-2\phi} \nabla^\mu e^\phi)$ under change of the sign in front of the dilaton field ϕ . In two dimensions $W[\phi] - W[-\phi]$ is a local functional of ϕ which can be expressed through the heat kernel coefficients. There are also higher dimensional generalizations of this relation. The dilaton ϕ can be replaced by a matrix-valued function. Apart from strings the dilaton interaction of the type described above appears also from the spherical reduction of higher dimensional theories [24].

Rather surprisingly, despite constant attention to this field of research, many problems are still left open. String models provide large number of applications for the spectral geometry methods which are the key subject of the present Conference.

Acknowledgements

I am grateful to M. Bordag, P. B. Gilkey, K. Kirsten, W. Kummer and A. Zelnikov for collaboration. This work has been supported in part by the DFG project Bo 1112/11-1.

References

- [1] O. Alvarez, Nucl. Phys. B **216** (1983) 125.
- [2] T. P. Branson and P. B. Gilkey, Commun. Part. Diff. Eqs. **15** (1990) 245.
- [3] D. V. Vassilevich, J. Math. Phys. **36** (1995) 3174.
- [4] T. P. Branson, P. B. Gilkey, K. Kirsten and D. V. Vassilevich, Nucl. Phys. B **563** (1999) 603.
- [5] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory. (Cambridge Univ. Press, 1987).
- [6] G. Grubb, Annali della Scuola Normale Superiore di Pisa **1** (1974) 1.
- [7] P.B. Gilkey and L. Smith, J. Diff. Geom. **15** (1983) 393.
- [8] D.M. McAvity and H. Osborn, Class. Quantum Grav. **8** (1991) 1445.
- [9] J.S. Dowker and K. Kirsten, Class. Quantum Grav. **14** (1997) L169; **16** (1999) 1917.
- [10] I. Avramidi and G. Esposito, Class. Quantum Grav. **15** (1998) 1141; Class. Quantum Grav. **15** (1998) 281; Commun. Math. Phys. **200** (1999) 495; Trends in Mathematical Physics: Proceedings. Edited by V. Alexiades and G. Siopsis. Cambridge, Mass., International Press, 1999. (AMS/IP Studies in Advanced Mathematics, v. 13). pp. 15-34.
- [11] H. Osborn, Nucl. Phys. B **363** (1991) 486.
- [12] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, Nucl. Phys. B **280** [FS **18**] (1987) 599; C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B **308** (1988) 221.
- [13] W. Kummer and D. V. Vassilevich, Open strings, Born-Infeld action and the heat kernel, hep-th/0001125.
- [14] W. Kummer and D. V. Vassilevich, JHEP **0007** (2000) 012.
- [15] H. Dorn and H. J. Otto, Z. Phys. C **32** (1986) 599; J. Dai, R. G. Leigh and J. Polchinski, Mod. Phys. Lett. A **4** (1989) 2073. R. G. Leigh, Mod. Phys. Lett. A **4** (1989) 2767.
- [16] D. V. Vassilevich, JHEP **0103** (2001) 023.
- [17] M. F. Atiyah, V. K. Patodi and I. M. Singer, Math. Proc. Cambridge Phil. Soc. **77** (1975) 43; **78** (1976) 405; **79** (1980) 71.
- [18] T. H. Buscher, Phys. Lett. B **194** (1987) 59; Phys. Lett. B **201** (1988) 466.

- [19] P. Gilkey, K. Kirsten, J. Park and D. Vassilevich, Asymptotics of the Heat Equation with exotic boundary conditions or with time dependent coefficients, this volume, math-ph/0105009.
- [20] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125** (1983) 136; L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690.
- [21] M. Bordag and D. V. Vassilevich, J. Phys. A **32** (1999) 8247; I. G. Moss, Phys. Lett. B **491** (2000) 203; P. B. Gilkey, K. Kirsten and D. V. Vassilevich, Nucl. Phys. B **601** (2001) 125; M. Bordag, D. Vassilevich, H. Falomir and E. M. Santangelo, Multiple reflection expansion and heat kernel coefficients, hep-th/0103037.
- [22] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, Solvable models of quantum mechanics (Springer, NY, 1988).
- [23] A. S. Schwarz and A. A. Tseytlin, Nucl. Phys. B **399** (1993) 691; D. V. Vassilevich and A. Zelnikov, Nucl. Phys. B **594** (2001) 501.
- [24] V. Mukhanov, A. Wipf and A. Zelnikov, Phys. Lett. B **332** (1994) 283; W. Kummer, H. Liebl and D. V. Vassilevich, Mod. Phys. Lett. A **12** (1997) 2683; W. Kummer and D. V. Vassilevich, Annalen Phys. **8** (1999) 801; Phys. Rev. D **60** (1999) 084021.